Electronic Companion: Supplementary Proofs for "The Choice Overload Effect in Online Recommender Systems"

In this document we provide supplementary proofs for Lemma 1, Lemma 2 and Proposition 3 mentioned in Appendices C-D of the paper.

Proof of Lemma 1. By Assumption 1, given any n, the deterministic part of v_i is decreasing in i for $1 \leq i \leq n$. Since the error terms ε_i's are i.i.d, v_i is stochastically decreasing in i. We also observe that V_n is stochastically increasing in n. Therefore, for $n \geq 2$, we have

$$\mathbb{E}[V_n] - \mathbb{E}[V_{n-1}] = \mathbb{E}[(v_n - V_{n-1})^+] \geq \mathbb{E}[(v_{n+1} - V_n)^+] = \mathbb{E}[V_{n+1}] - \mathbb{E}[V_n].$$

The inequality holds because v_n is stochastically decreasing in n and V_n is stochastically increasing in n. Thus $\mathbb{E}[V_n]$ is increasing concave in n.

Moreover, we have $\mathbb{E}[V_n] - \mathbb{E}[V_{n-1}] = \mathbb{E}[(v_n - V_{n-1})^+] = \mathbb{E}[(v_n - V_{n-1}) \cdot 1_{\{v_n \geq V_{n-1}\}}]$. Let $X_n = v_n - V_{n-1}$ and $F_n(\cdot)$ be the cumulative distribution of X_n. Then we have $\mathbb{E}[X_n] = \mathbb{E}[v_n] - \mathbb{E}[V_{n-1}] \leq \mu_n - \mathbb{E}[\max\{v_0, v_1\}] < \infty$. Let $v_n^1, v_n^2, \ldots, v_n^n$ be n i.i.d. random variables that follow the probability distribution of v_n. We then have

$$\mathbb{E}[1_{\{v_n \geq V_{n-1}\}}] \leq \mathbb{E}[1_{\{v_n \geq \max\{v_1, v_2, \ldots, v_{n-1}\}\}}] \leq \mathbb{E}[1_{\{v_n^n \geq \max\{v_n^1, v_n^2, \ldots, v_n^{n-1}\}\}}] = \frac{1}{n}. \tag{EC.1}$$

The second inequality in Equation (EC.1) holds since $v_i = \mu_i + \epsilon_i$ for $i = 1, \ldots, n$ where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and ϵ_i's are i.i.d. random variables, and $v_n^1, v_n^2, \ldots, v_n^n$ have the same probability distribution as v_n. The last equality in Equation (EC.1) follows by symmetry. Therefore, we have $\mathbb{E}[(v_n - V_{n-1}) \cdot 1_{\{v_n \geq V_{n-1}\}}] \leq \int_{F_n^{-1}(1-\frac{1}{n})}^{\infty} x dF_n(x)$. Since $\mathbb{E}[X_n]$ is finite, $\int_{F_n^{-1}(1-\frac{1}{n})}^{\infty} x dF_n(x) \to 0$ as $n \to \infty$. Therefore, $\mathbb{E}[V_n] - \mathbb{E}[V_{n-1}] \to 0$ as $n \to \infty$. ∎

Proof of Lemma 2. The proof is by contradiction. Suppose $S_D^* \neq \{1, 2, \ldots, d\}$ for any $1 \leq d \leq |S|$. In other words, there exists m, n such that $m \in S_D^*$, $n \in S \backslash S_D^*$, and $\mu_m < \mu_n$. Construct two new consideration sets: $S_D^- = S_D^* \backslash \{m\}$ and $S_D' = (S_D^* \backslash \{m\}) \cup \{n\}$. We can show that either $\mathbb{E}U(S_D^*) < \mathbb{E}U(S_D')$ or $\mathbb{E}U(S_D^*) < \mathbb{E}U(S_D^-)$, which contradicts with the optimality of S_D^*.

Let $Y(S_D)$ be the highest utility of selecting an option from S_D^+, i.e., $Y(S_D) = \max_{i \in S_D^+} \tilde{v}_i(S_D)$, where $\tilde{v}_i(S_D) = v_i - \beta \mathbb{E}\left[(\max_{j \in S^+} v_j - v_i)^+ \middle| \{v_k : k \in S_D^+\}\right]$ for $i \in S_D$ and $\tilde{v}_0(S_D) = v_0$. The proof consists of three steps.

Step 1 For $S_D \subsetneq S$, define $\hat{v}_i(S_D)$ as

$$\hat{v}_i(S_D) = \begin{cases} v_i - \beta \mathbb{E}\left[(\max_{j \in S \backslash S_D} v_j - v_i)^+ \middle| v_i\right], & i \neq 0, \\ v_0, & i = 0. \end{cases}$$

First, we will show that $Y(S_D) = \max_{i \in S_D^+} \hat{v}_i(S_D)$. Let $i' =_{i \in S_D} \tilde{v}_i(S_D)$ denote the option in S_D that yields the highest utility, so $\tilde{v}_{i'}(S_D) \geq \tilde{v}_i(S_D)$ for all $i \in S_D$. We can show that $v_{i'} \geq v_i$ and $\hat{v}_{i'}(S_D) \geq \hat{v}_i(S_D)$ for all $i \in S_D$. In other words, $i' =_{i \in S_D} v_i =_{i \in S_D} \hat{v}_i(S_D)$. To see this, observe that if $i \in S_D$, then given $\{v_k : k \in S_D^+, k \neq i\}$, both $\tilde{v}_i(S_D)$ and $\hat{v}_i(S_D)$ are strictly increasing (and hence order-preserving functions) in v_i.

Now we will prove $Y(S_D) = \max_{i \in S_D^+} \hat{v}_i(S_D)$ by showing that it holds in both cases of $v_0 \leq v_{i'}$ and $v_0 > v_{i'}$:

Case 1: If $v_0 \leq v_{i'}$, then $v_{i'} \geq v_i$ for all $i \in S_D^+$ and thus $(\max_{j \in S^+} v_j - v_{i'})^+ = (\max_{j \in S/S_D} v_j - v_{i'})^+$. Then we have $\tilde{v}_{i'}(S_D) = \hat{v}_{i'}(S_D)$. Therefore, $Y(S_D) = \max_{i \in S_D^+} \tilde{v}_i(S_D) = \max\{v_0, \tilde{v}_{i'}(S_D)\} = \max\{v_0, \hat{v}_{i'}(S_D)\} = \max_{i \in S_D^+} \hat{v}_i(S_D)$ (The last equality holds since $i' =_{i \in S_D} \hat{v}_i(S_D)$). In particular, if a consumer searches $S_D \subsetneq S$ and finds it optimal to make a purchase, i.e., $i^* = i'$ where i^* represents the optimal choice, we have $Y(S_D) = \tilde{v}_{i^*}(S_D) = \hat{v}_{i^*}(S_D)$.

Case 2: If $v_0 > v_{i'}$, then $\tilde{v}_0(S_D) = v_0 > v_{i'} \geq \tilde{v}_{i'}(S_D)$, and thus $Y(S_D) = \tilde{v}_0(S_D) = v_0$. Notice that we also have $\hat{v}_0(S_D) = v_0 > v_{i'} \geq \hat{v}_{i'}(S_D)$. Since $i' =_{i \in S_D} \hat{v}_i(S_D)$, we have $\max_{i \in S_D^+} \hat{v}_i(S_D) = \hat{v}_0(S_D) = v_0$. Therefore, $Y(S_D) = v_0 = \max_{i \in S_D^+} \hat{v}_i(S_D)$.

Step 2 Next, we will prove that $Y(S_D^*) \preceq Y(S_D')$. First, we observe that for $i \in S_D^* \backslash \{m\}$, we have $\hat{v}_i(S_D^*) \preceq \hat{v}_i(S_D')$, i.e.,

$$v_i - \beta \mathbb{E}\left[(\max_{j \in S \backslash S_D^*} v_j - v_i)^+ \middle| v_i \right] \preceq v_i - \beta \mathbb{E}\left[(\max_{j \in S \backslash S_D'} v_j - v_i)^+ \middle| v_i \right].$$

The proof for the above stochastic inequality is as follows: Observe that $S \backslash S_D'$ is obtained by replacing option n in $S \backslash S_D^*$ with option m. That is, define $S_N := (S \backslash S_D^*) \backslash \{n\}$, then $S \backslash S_D^* = S_N \cup \{n\}$ and $S \backslash S_D' = S_N \cup \{m\}$. Notice that $\mu_m < \mu_n$ implies $v_m \prec v_n$. Then given any value v, we have $(\max_{j \in S_N \cup \{n\}} v_j - v)^+ \succeq (\max_{j \in S_N \cup \{m\}} v_j - v)^+$, i.e., $(\max_{j \in S \backslash S_D^*} v_j - v)^+ \succeq (\max_{j \in S \backslash S_D'} v_j - v)^+$, and thus $\mathbb{E}\left[(\max_{j \in S \backslash S_D^*} v_j - v)^+\right] \geq \mathbb{E}\left[(\max_{j \in S \backslash S_D'} v_j - v)^+\right]$. Since $\beta > 0$, we have $v - \beta \mathbb{E}\left[(\max_{j \in S \backslash S_D^*} v_j - v)^+\right] \leq v - \beta \mathbb{E}\left[(\max_{j \in S \backslash S_D'} v_j - v)^+\right]$ for any value v, which implies $\hat{v}_i(S_D^*) \preceq \hat{v}_i(S_D')$.

For $i = m$, since $v_m \prec v_n$, we have

$$v_m - \beta \mathbb{E}\left[(\max_{j \in (S \backslash S_D^*} v_j - v_m)^+ \middle| v_m \right] \prec v_n - \beta \mathbb{E}\left[(\max_{j \in S \backslash S_D^*} v_j - v_n)^+ \middle| v_n \right] \preceq v_n - \beta \mathbb{E}\left[\max_{j \in S \backslash S_D'} (v_j - v_n)^+ \middle| v_n \right], \tag{EC.2}$$

i.e., $\hat{v}_m(S_D^*) \prec \hat{v}_n(S_D')$.

Since $\hat{v}_i(S_D^*) \preceq \hat{v}_i(S_D')$ for all $i \in S_D^* \backslash \{m\}$ (or $i \in S_D' \backslash \{n\}$ equivalently) and $\hat{v}_m(S_D^*) \prec \hat{v}_n(S_D')$, it follows that $Y(S_D^*) \preceq Y(S_D')$.

Step 3 Finally, we will show that either $\mathbb{E}U(S_D^*) < \mathbb{E}U(S_D')$ or $\mathbb{E}U(S_D^*) < \mathbb{E}U(S_D^-)$. We do so by considering two possible cases, depending on whether $Y(S_D^*) \prec Y(S_D')$ or $Y(S_D^*) \overset{d}{=} Y(S_D')$ holds (recall from Step 2 that $Y(S_D^*) \preceq Y(S_D')$), where $\overset{d}{=}$ denotes equality in distribution.

Case 1: If $Y(S_D^*) \prec Y(S_D')$, then $\mathbb{E}Y(S_D^*) < \mathbb{E}Y(S_D')$. Since $c > 0$ and $|S_D^*| = |S_D'|$, we have

$$\mathbb{E}U(S_D^*) = \mathbb{E}Y(S_D^*) - c|S_D^*| < \mathbb{E}Y(S_D') - c|S_D'| = \mathbb{E}U(S_D').$$

Case 2: If $Y(S_D^*) \overset{d}{=} Y(S_D')$, then $\Pr(Y(S_D^*) \leq y) = \Pr(Y(S_D') \leq y)$ for any y. Notice that for any S_D, $\hat{v}_i(S_D)$'s, where $i \in S_D$, are independent of each other due to the independence of ϵ_i's. Since $S_D^* = S_D^- \cup \{m\}$ and $S_D' = S_D^- \cup \{n\}$, we have, for any value y,

$$\Pr(Y(S_D^*) \leq y) = \Pr(Y(S_D^-) \leq y) \cdot \Pr(\hat{v}_m(S_D^*) \leq y),$$
$$\Pr(Y(S_D') \leq y) = \Pr(Y(S_D^-) \leq y) \cdot \Pr(\hat{v}_n(S_D') \leq y).$$

Since $\hat{v}_m(S_D^*) \prec \hat{v}_n(S_D')$, the set $X := \{x : \Pr(\hat{v}_m(S_D^*) \leq x) > \Pr(\hat{v}_n(S_D') \leq x)\}$ is nonempty. Therefore, $\Pr(Y(S_D^-) \leq x) = 0$ for all $x \in X$. This implies $\Pr(Y(S_D^*) \leq x) = \Pr(Y(S_D') \leq x) = 0$ for all $x \in X$, and hence $\sup X$ is finite. We now show that $\Pr(Y(S_D^*) \leq y) = \Pr(Y(S_D^-) \leq y)$ for any y, i.e., $Y(S_D^*) \overset{d}{=} Y(S_D^-)$.

In this case, given that $\sup X$ is finite, there must exists $\bar{x} \in X$ such that $\Pr(\hat{v}_m(S_D^*) \leq \bar{x}) = 1$ while $\Pr(\hat{v}_n(S_D') \leq \bar{x}) < 1$ (see proof of this Claim below). Therefore, if $y \leq \bar{x}$, we have $\Pr(Y(S_D^-) \leq y) \leq \Pr(Y(S_D^-) \leq \bar{x}) = 0$, and thus $\Pr(Y(S_D^*) \leq y) = \Pr(Y(S_D^-) \leq y) = 0$. If $y > \bar{x}$, $\Pr(\hat{v}_m(S_D^*) \leq y) = \Pr(\hat{v}_m(S_D^*) \leq \bar{x}) = 1$, and we again have $\Pr(Y(S_D^*) \leq y) = \Pr(Y(S_D^-) \leq y)$. Therefore, $Y(S_D^*) \overset{d}{=} Y(S_D^-)$, and thus $\mathbb{E}Y(S_D^*) = \mathbb{E}Y(S_D^-)$. Since $c > 0$ and $|S_D^*| > |S_D^-|$, we have

$$\mathbb{E}U(S_D^*) = \mathbb{E}Y(S_D^*) - c|S_D^*| < \mathbb{E}Y(S_D^-) - c|S_D^-| = \mathbb{E}U(S_D^-).$$

To prove the Claim above, let $g(\epsilon) = \mu_m + \epsilon - \beta\mathbb{E}[(\max_{j \in S \setminus S_D^*} v_j - \mu_m - \epsilon)^+ | \epsilon]$, which is a strictly increasing function in ϵ. Denote $F(\cdot)$ as the cumulative distribution function of ϵ_i's. We then have $\Pr(\hat{v}_m(S_D^*) \leq x) = F(g^{-1}(x))$, and $\Pr(\hat{v}_n(S_D') \leq x) \leq \Pr(\hat{v}_n(S_D^*) \leq x) = F(g^{-1}(x) - \Delta)$ (the inequality follows from Equation (EC.2)), where $\Delta := \mu_n - \mu_m > 0$. Thus, given that $\sup X$ is finite, there must exists $\bar{x} \in X$ such that $\Pr(\hat{v}_m(S_D^*) \leq \bar{x}) = F(f^{-1}(\bar{x})) = 1$ while $\Pr(\hat{v}_n(S_D') \leq \bar{x}) \leq F(f^{-1}(\bar{x}) - \Delta) < 1$.

Combining the two cases, either $\mathbb{E}U(S_D^*) < \mathbb{E}U(S_D')$ or $\mathbb{E}U(S_D^*) < \mathbb{E}U(S_D^-)$ holds, which contradicts with the optimality of S_D^*. This concludes the proof. ■

Proof of Proposition 3. Given the choice set $S_n = \{1, 2, ..., n\}$, the consumer's purchase probability can be written as

$$\omega(S_n) = \psi(S_n) \cdot \Pr(\max_{i \in S_D^*(S_n)} \tilde{v}_i(S_D^*(S_n)) \geq v_0),$$

where we have shown in Proposition 2 that for $\beta > 0$, as n increases, $\psi(S_n)$ increases when $n \leq \bar{n}$ and decreases otherwise. First, we observe that the threshold $\bar{n} =_n U_{n,n}$ does not depend on the value of β. We consider two scenarios, depending on whether $n < \bar{n}$.

(I) When $n < \bar{n}$: we have $\psi(S_{n+1}) > \psi(S_n)$ for all $n < \bar{n}$. We have shown that the consumer prefers full search to partial search for all $n \leq \bar{n}$. Then comparing the purchase probability conditional on search for S_n and S_{n+1} for all $n < \bar{n}$, we have $\Pr(\max_{i \in S_{n+1}} v_i \geq v_0) \geq \Pr(\max_{i \in S_n} v_i \geq v_0)$ since $S_n \subset S_{n+1}$. It follows that $\omega(S_{n+1}) > \omega(S_n)$ for all $n \leq \bar{n}$.

(II) When $n \geq \bar{n}$: we have $\psi(S_{n+1}) \leq \psi(S_n)$. Denote the size of the optimal consideration set as $\tau(n)$. That is, for choice set S_n we have $S_D^*(S_n) = \{1, 2, ..., \tau(n)\} = S_{\tau(n)}$. Notice that when $\psi(S_{n+1}) \leq \psi(S_n)$, $\omega(S_{n+1}) \leq \omega(S_n)$ holds if

$$\Pr(\max_{i \in S_{\tau(n+1)}} \hat{v}_i(n+1, \tau(n+1)) \geq v_0) \leq \Pr(\max_{i \in S_{\tau(n)}} \hat{v}_i(n, \tau(n)) \geq v_0),$$

where $\hat{v}_i(n, m)$ for $m \leq n$ is as defined in Proposition 2. A sufficient condition for the above inequality to hold is $\tau(n+1) \leq \tau(n)$ for all $n \geq \bar{n}$. We will show that there exists $\bar{\beta}$ such that the above condition holds when $\beta < \bar{\beta}$. The proof consists of four steps:

Step 1 Let $\Delta\hat{v}_i(n, m) = \hat{v}_i(n, m) - \hat{v}_i(n+1, m)$ for $m \leq n$. We will first show three results regarding $\Delta\hat{v}_i(n, m)$: (1) $\Delta\hat{v}_i(n, m) \geq 0$ for all $i \in S_m^+$, (2) $\Delta\hat{v}_i(n, m)$ is stochastically increasing in i for $i \in S_m^+$, and (3) given any $m^* < n$ and $i \in S_{m^*}^+$, $\Delta\hat{v}_i(n, m)$ is stochastically increasing in m for $m^* \leq m \leq n$.

Define $V_{n\backslash m} = \max_{j\in S_n\backslash S_m} v_j$. Then when $m < n$, for $i \in S_m$,

$$\begin{aligned}\Delta\hat{v}_i(n,m) &= \beta\mathbb{E}\left[(\max_{j\in S_{n+1}\backslash S_m} v_j - v_i)^+ - (\max_{j\in S_n\backslash S_m} v_j - v_i)^+ \middle| v_i\right] \\ &= \beta\mathbb{E}\left[[(v_{n+1} - v_i)^+ - (V_{n\backslash m} - v_i)^+]\cdot \mathbf{1}_{\{v_{n+1} > V_{n\backslash m}\}} \middle| v_i\right].\end{aligned}$$

When $m = n$, for $i \in S_m$,

$$\Delta\hat{v}_i(n,m) = \beta\mathbb{E}\left[(v_{n+1} - v_i)^+ \middle| v_i\right].$$

We also have $\Delta\hat{v}_0(n,m) = 0$ for all $m \le n$. Then result (1) holds obviously when $m = n$ given $\beta > 0$. It also holds when $m < n$, because $\beta > 0$ and $\max_{j\in S_n\backslash S_m} v_j \preceq \max_{j\in S_{n+1}\backslash S_m} v_j$. To prove result (2), we first notice that $\Delta\hat{v}_i(n,m)$ is stochastically increasing in i for $i \in S_m$ given any $m \le n$, because v_i is stochastically decreasing in i and $\Delta\hat{v}_i(n,m)$ is a decreasing function of the value of v_i. To see the latter, notice that $g_1(x) := (a-x)^+ - (b-x)^+$ is decreasing in x for any $a > b$ and $g_2(x) := (c-x)^+$ is also decreasing in x for any c. Since $\Delta\hat{v}_0(n,m) = 0$ while $\Delta\hat{v}_i(n,m) \ge 0$ for $i \in S_m$, we conclude that $\Delta\hat{v}_i(n,m)$ is stochastically increasing in i for $i \in S_m^+$ given any $m \le n$. As for result (3), since $V_{n\backslash m}$ is stochastically decreasing in m, it follows that given any value of v_i where $i \in S_{m^*}$, $\Delta\hat{v}_i(n,m)$ is increasing in m for $m^* \le m < n$. Moreover,

$$\Delta\hat{v}_i(n,n) - \Delta\hat{v}_i(n,n-1) = \beta\mathbb{E}\left[(v_{n+1} - v_i)^+ - [(v_{n+1} - v_i)^+ - (v_n - v_i)^+]\cdot \mathbf{1}_{\{v_{n+1} > v_n\}} \middle| v_i\right] \ge 0.$$

Also notice that $\Delta\hat{v}_0(n,m) = 0$ for all $m \le n$. Therefore, for any $i \in S_{m^*}^+$, $\Delta\hat{v}_i(n,m)$ is stochastically increasing in m for $m^* \le m \le n$.

Step 2 For any $m \le n$, define $\hat{V}(n,m) = \max_{i\in S_m^+} \hat{v}_i(n,m)$, which represents the maximum utility of an option in S_m^+ given the choice set S_n. Let $\Delta\hat{V}(n,m) = \hat{V}(n,m+1) - \hat{V}(n,m)$ for $m < n$. We will next show that for any given m, $\mathbb{E}[\Delta\hat{V}(n,m)]$ is decreasing in n, i.e., $\mathbb{E}[\Delta\hat{V}(n+1,m)] - \mathbb{E}[\Delta\hat{V}(n,m)] \le 0$ for all $n > m$.

Following the notation in Lemma 2, we define $i' =_{i\in S_m} \hat{v}_i(n,m)$. By proof of Lemma 2, we have $i' =_{i\in S_m} v_i$, i.e., i' does not depend on n. Define $\bar{V}(n,m) = \max_{i\in S_m} \hat{v}_i(n,m)$ for $m \le n$, so $\hat{V}(n,m) = \max\{v_0, \bar{V}(n,m)\}$. Given any n, we have $\bar{V}(n,m) \preceq \bar{V}(n,m+1)$ for $m < n$, because $\hat{v}_i(n,m)$ is stochastically increasing in m and $S_m \subset S_{m+1}$. Define $\Delta\bar{V}(n,m) = \bar{V}(n,m) - \bar{V}(n+1,m)$ for $m \le n$, then we have

$$\begin{aligned}\Delta\bar{V}(n,m) &= \max_{i\in S_m} \hat{v}_i(n,m) - \max_{i\in S_m} \hat{v}_i(n+1,m) \\ &= \max_{i\in S_m}\Big(\hat{v}_i(n+1,m) + \Delta\hat{v}_i(n,m)\Big) - \max_{i\in S_m} \hat{v}_i(n+1,m) \\ &= \sum_{i\in S_m} \Delta\hat{v}_i(n,m)\cdot \Pr(i' = i).\end{aligned}$$

Since $\Delta\hat{v}_i(n,m)$ is stochastically increasing in i (result (2) in Step 1) and also stochastically increasing in m (result (3) in Step 1), it follows that $\Delta\bar{V}(n,m)$ is stochastically increasing in m given any n, i.e., $\Delta\bar{V}(n,m) \preceq \Delta\bar{V}(n,m+1)$ for $m < n$. Moreover, $\Delta\bar{V}(n,m) \ge 0$ because $\Delta\hat{v}_i(n,m) \ge 0$ for $i \in S_m$.

Notice that $\hat{V}(n,m) = \max\{v_0, \bar{V}(n,m)\}$, and thus we have

$$
\begin{aligned}
&\mathbb{E}[\Delta\hat{V}(n+1,m)] - \mathbb{E}[\Delta\hat{V}(n,m)] \\
=&\mathbb{E}\left[\hat{V}(n+1,m+1) - \hat{V}(n+1,m)\right] - \mathbb{E}\left[\hat{V}(n,m+1) - \hat{V}(n,m)\right] \\
=&\mathbb{E}\left[\hat{V}(n,m) - \hat{V}(n+1,m)\right] - \mathbb{E}\left[\hat{V}(n,m+1) - \hat{V}(n+1,m+1)\right] \\
=&\mathbb{E}\left[\max\{v_0,\bar{V}(n,m)\} - \max\{v_0,\bar{V}(n+1,m)\}\right] - \mathbb{E}\left[\max\{v_0,\bar{V}(n,m+1)\} - \max\{v_0,\bar{V}(n+1,m+1)\}\right] \\
=&\mathbb{E}\left[\max\{v_0,\bar{V}(n+1,m)+\Delta\bar{V}(n,m)\} - \max\{v_0,\bar{V}(n+1,m)\}\right] \\
&-\mathbb{E}\left[\max\{v_0,\bar{V}(n+1,m+1)+\Delta\bar{V}(n,m+1)\} - \max\{v_0,\bar{V}(n+1,m+1)\}\right] \\
=&\mathbb{E}\left[\left(\bar{V}(n+1,m)+\Delta\bar{V}(n,m)-v_0\right)^+ - \left(\bar{V}(n+1,m)-v_0\right)^+\right] \\
&-\mathbb{E}\left[\left(\bar{V}(n+1,m+1)+\Delta\bar{V}(n,m+1)-v_0\right)^+ - \left(\bar{V}(n+1,m+1)-v_0\right)^+\right] \\
\le&\mathbb{E}\left[\left(\bar{V}(n+1,m)+\Delta\bar{V}(n,m+1)-v_0\right)^+ - \left(\bar{V}(n+1,m)-v_0\right)^+\right] \\
&-\mathbb{E}\left[\left(\bar{V}(n+1,m+1)+\Delta\bar{V}(n,m+1)-v_0\right)^+ - \left(\bar{V}(n+1,m+1)-v_0\right)^+\right] \\
\le&0.
\end{aligned}
$$

The first inequality follows by $\Delta\bar{V}(n,m) \preceq \Delta\bar{V}(n,m+1)$. The second inequality holds because $\bar{V}(n+1,m) \preceq \bar{V}(n+1,m+1)$ and $h(x) := (x+d-v)^+ - (x-v)^+$ is increasing in x given any values (d,v) where $d \ge 0$. We conclude that $\mathbb{E}[\Delta\hat{V}(n+1,m)] - \mathbb{E}[\Delta\hat{V}(n,m)] \le 0$, i.e., $\mathbb{E}[\Delta\hat{V}(n,m)]$ is decreasing in n.

Step 3 Now we will show that if $\tau(\bar{n}+1) < \bar{n}+1$, then $\tau(n)$ is weakly decreasing in n for all $n \ge \bar{n}$, i.e., $\tau(n+1) \le \tau(n)$ for all $n \ge \bar{n}$. By optimality of $\tau(\bar{n}+1)$, we have $\mathbb{E}\hat{V}(\bar{n}+1,m) - \mathbb{E}\hat{V}(\bar{n}+1,\tau(\bar{n}+1)) < c(m-\tau(\bar{n}+1))$ for all $m > \tau(\bar{n}+1)$. Then for $n = \bar{n}+2$, for all $m > \tau(\bar{n}+1)$, we have

$$
\begin{aligned}
\mathbb{E}\hat{V}(\bar{n}+2,m) - \mathbb{E}\hat{V}(\bar{n}+2,\tau(\bar{n}+1)) &= \sum_{k=\tau(\bar{n}+1)}^{m-1} \mathbb{E}[\Delta\hat{V}(\bar{n}+2,k)] \\
&\le \sum_{k=\tau(\bar{n}+1)}^{m-1} \mathbb{E}[\Delta\hat{V}(\bar{n}+1,k)] \\
&= \mathbb{E}\hat{V}(\bar{n}+1,m) - \mathbb{E}\hat{V}(\bar{n}+1,\tau(\bar{n}+1)) \\
&< c(m-\tau(\bar{n}+1)).
\end{aligned}
$$

Therefore, when the choice set size is $\bar{n}+2$, the utility of searching more than $\tau(\bar{n}+1)$ products can never be higher than that of searching $\tau(\bar{n}+1)$ products, then $\tau(\bar{n}+2) \le \tau(\bar{n}+1) < \bar{n}+2$. Then by induction and $\tau(\bar{n}+1) \le \bar{n} = \tau(\bar{n})$, we have $\tau(n+1) \le \tau(n)$ for all $n \ge \bar{n}$.

Step 4 Finally we will show that there exists $\bar{\beta}$ such that $\tau(n+1) \le \tau(n)$ for all $n \ge \bar{n}$ when $\beta < \bar{\beta}$. Given the result in Step 3, it is sufficient to find β such that $\tau(\bar{n}+1) < \bar{n}+1$. This is equivalent to

$$
\mathbb{E}\max_{i\in S_m^+} \hat{v}_i(\bar{n}+1,m) - cm > \mathbb{E}\max_{i\in S_{\bar{n}+1}^+} v_i - c(\bar{n}+1), \text{ for some } m \le \bar{n}, \tag{EC.3}
$$

where $\hat{v}_i(\bar{n}+1,m) = v_i - \mathbf{1}_{i\neq 0}\cdot\beta\mathbb{E}\left[(\max_{j\in S_{\bar{n}+1}\setminus S_m} v_j - v_i)^+ \middle| v_i\right]$. Notice that when $\beta = 0$, $\hat{v}_i(\bar{n}+1,m) = v_i$, and condition (EC.3) holds for $m = \bar{n}$ (because $U_{\bar{n}+1,\bar{n}} = U_{\bar{n},\bar{n}} > U_{\bar{n}+1,\bar{n}+1}$ by definition of $\bar{n}$ as shown in the proof of Proposition 2). Moreover, $\mathbb{E}\max_{i\in S_m^+}\hat{v}_i(\bar{n}+1,m)-cm$ is continuous and decreasing in β for all $m\leq\bar{n}$. Therefore, there must exists $\bar{\beta}>0$ such that condition (EC.3) holds when $\beta<\bar{\beta}$. The threshold $\bar{\beta}$ can be obtained by solving for the unique $\beta=\beta_m$, for each $m\leq\bar{n}$, such that:

$$\mathbb{E}\max_{i\in S_m^+}\hat{v}_i(\bar{n}+1,m)-cm=\mathbb{E}\max_{i\in S_{\bar{n}+1}^+}v_i-c(\bar{n}+1),$$

and taking $\bar{\beta}=\max_m\beta_m$. ■